SECTION 17.5: DIVERGENCE AND CURL

RECALL: Given a vector field $\vec{F} = \langle M, N \rangle$, the **divergence** of $\vec{F}$ is the function $M_x + N_y$. Abusing notation:

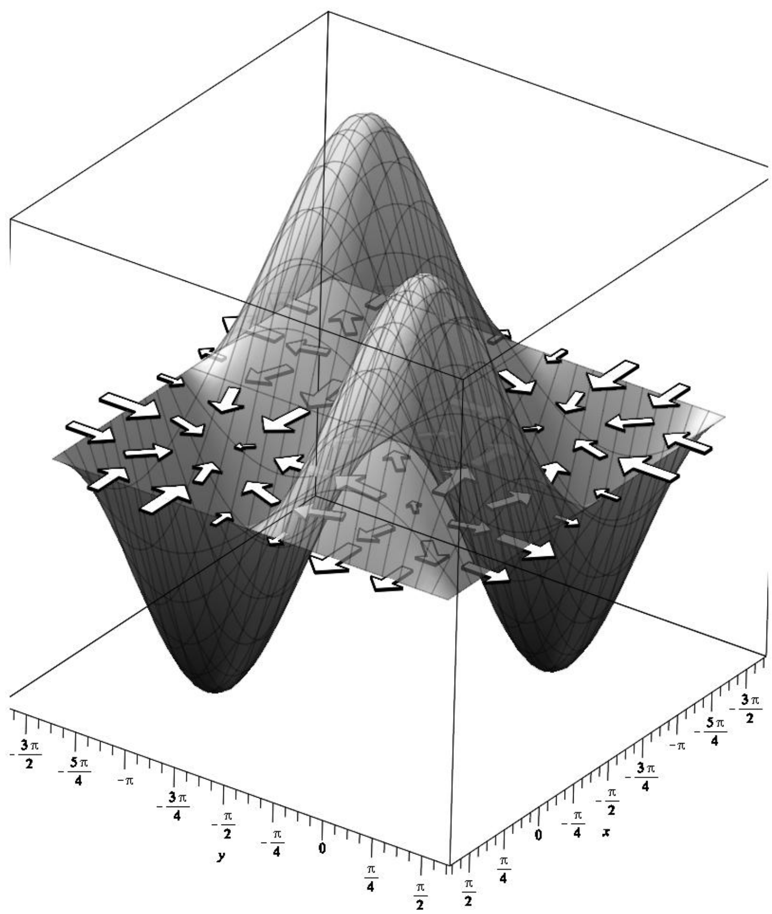
$$M_x + N_y = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle M, N \rangle$$

We define $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$ and re-define the divergence of $\vec{F}$, $\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$.

Note that the divergence of a vector field is a **scalar** function.

The divergence of a vector field measures the tendency for a vector field to 'spread out.'

Below is the sketch of a 2-D vector field with its divergence. Divergence is high (positive) where the arrows are 'spreading out.' Points where the divergence is positive are called **sources**. Divergence is low (negative) when the arrows are coming together. Points where divergence is negative are called **sinks**.



GREEN'S THEOREM (FLUX FORM REPRISE): Outward Flux $= \oint_C \vec{F} \cdot d\vec{n} = \iint_R \nabla \cdot \vec{F} \, dA$

Hence: **positive** divergence corresponds to **outward** flux; **negative** divergence corresponds to **inward** flux.

EXAMPLE 1: Let $\vec{F}(x,y) = \left\langle x^2 - y^2 - 4, 2xy \right\rangle$.

1. Find $\operatorname{div} \vec{F}(x,y)$.

Ans: $\operatorname{div} \vec{F}(x,y) = 4x$

2. Find $\operatorname{div} \vec{F}(-2,0)$. Is $(-2,0)$ a source or a sink?

Ans: $\operatorname{div} \vec{F}(-2,0) = -8$; sink.

3. Find $\operatorname{div} \vec{F}(2,0)$. Is $(2,0)$ a source or a sink?

Ans: $\operatorname{div} \vec{F}(2,0) = 8$; source.

4. Graph $\vec{F}(x,y)$ near $(-2,0)$ and again near $(2,0)$ to check the reasonableness of your results.

EXAMPLE 2: Suppose $\vec{F} = \langle M, N, P \rangle$. How would you generalize ∇ and $\operatorname{div} \vec{F}$ to three dimensions?

EXAMPLE 3: Graph each of the following fields and compute $\operatorname{div} \vec{F}$.

1. (Radial Flow) $\vec{F}(x,y,z) =< x, y, z >$

Ans: $\operatorname{div} \vec{F}(x,y,z) = 3$

2. (Spiral Flow) $\vec{F}(x,y,z) =< -y, x, z >$

Ans: $\operatorname{div} \vec{F}(x,y,z) = 1$

DEFINITION: A field is called, **source free** (**divergence free**, **solenoidal**, **incompressible**) if $\nabla \cdot \vec{F} = 0$.

EXAMPLE 4: Compute div $\vec{F}$ to show each field below is source free.

1. (Rotation Field) $\vec{F}(x, y, z) =< 2z - 3y, 3x - z, y - 2x >$

2. (General Rotation Field) $\vec{F}(x, y, z) = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle =< a_2z - a_3y, a_3x - a_1z, a_1y - a_2x >$

3. $\vec{F}(x, y, z) =< f(y, z), g(x, z), h(x, y) >$

RECALL: Given a vector field $\vec{F}(x, y) = \langle M(x, y), N(x, y)\rangle$, the **scalar curl** of $\vec{F}$ is the function $N_x - M_y$.

To generalize this, note that we may regard $\vec{F}(x, y) = \langle M(x, y), N(x, y), 0\rangle$. With $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$:

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle M(x, y), N(x, y), 0\rangle = \langle 0, 0, N_x - M_y\rangle$$

Hence the scalar curl of $\vec{F}$ in this case is $(\nabla \times \vec{F}) \cdot \hat{\mathrm{k}}$.

GREEN'S THEOREM (CIRCULATION FORM REPRISE): Circulation $= \oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \hat{\mathrm{k}}\, dA$

Positive curl corresponds to **counter-clockwise** rotation; **negative** curl corresponds to **clockwise** rotation.

EXAMPLE 5: Let $\vec{F}(x, y) = \left\langle x^2 - y^2 - 4, 2xy \right\rangle$.

1. Find the scalar curl of $\vec{F}(x, y)$.

Ans: scalar curl is $4y$.

2. Find the scalar curl of $\vec{F}(x, y)$ at $(0, -2)$. Is the flow near $(0, -2)$ clockwise or counter-clockwise?

Ans: -8; clockwise rotation

3. Find the scalar curl of $\vec{F}(x, y)$ at $(0, 2)$. Is the flow near $(0, 2)$ clockwise or counter-clockwise?

Ans: 8; counter-clockwise rotation

4. Graph $\vec{F}(x, y)$ near $(0, -2)$ and again near $(0, 2)$ to check the reasonableness of your results.

DEFINITION: If $\vec{F} = \langle M, N, P \rangle$ then the curl of $\vec{F}$, curl $\vec{F} = \nabla \times \vec{F}$. If curl $\vec{F} = \vec{0}$, $\vec{F}$ is called **irrotational**.

EXAMPLE 6: Graph each of the following fields and compute curl $\vec{F}$.

1. (Radial Flow) $\vec{F}(x, y, z) =< x, y, z >$

Ans: curl $\vec{F}(x, y, z) = \langle 0, 0, 0 \rangle$

2. (Spiral Flow) $\vec{F}(x, y, z) =< -y, x, z >$

Ans: curl $\vec{F}(x, y, z) = \langle 0, 0, 2 \rangle$

EXAMPLE 7: Show curl $\vec{F} = 2\vec{a}$ for a general rotation field:

$$\vec{F}(x, y, z) = \vec{a} \times \vec{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle =< a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x >$$

EXAMPLE 8: Show that if $\vec{F} = \langle M, N, P \rangle$ then $\nabla \times \vec{F} = \vec{0}$ is equivalent to $M_y = N_x$, $M_z = P_x$ and $N_z = P_y$.

NOTE: Hence, under suitable conditions, **irrotational** fields are **conservative**!

IMPORTANT IDENTITIES

EXAMPLE 9: Prove the following identities:

1. If $f(x, y, z)$ has continuous second partials, then $\nabla \times (\nabla f) = \vec{0}$.

2. If $\vec{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$ where M, N, and P have continuous second partials, then

$$\text{div}(\text{curl}\, \vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

HOMEWORK: Section 17.5: 9 - 49 every other odd.

BONUS TRACKS: Suppose the components of $\vec{F} = \langle M, N, P \rangle$ have continuous first partials throughout an open simply connected region in space.

- We know that $\vec{F}$ is conservative is equivalent to $\nabla \times \vec{F} = \vec{0}$. Said differently, $\nabla \times \vec{F} = \vec{0}$ guarantees a function ϕ so that $\vec{F} = \nabla \phi$. We call ϕ a **potential** for $\vec{F}$. Hence, $\nabla \times \vec{F} = \nabla \times (\nabla \phi) = \vec{0}$.
- It turns out that under suitable conditions, if $\nabla \cdot \vec{F} = 0$, then there is a field $\vec{G}$ so that $\vec{F} = \nabla \times \vec{G}$. In this case, $\vec{G}$ is called a **vector potential** for $\vec{F}$. Hence, $\nabla \cdot \vec{F} = \nabla \cdot (\nabla \times \vec{G}) = 0$.
- **HELMHOLTZ DECOMPOSITION THEOREM:**

  Under suitable conditions, a vector field $\vec{F}$ can be written as $\vec{F} = \nabla \phi + \nabla \times \vec{G}$.

  That is, $\vec{F}$ is the sum of a conservative (irrotational) field and a source free (solenoidal) field!